ON FIBONACCI ZETA VALUES

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Suppose that $\alpha,\ \beta \in \mathbb{C}$ satisfy $|\beta| < 1$ and $\alpha\beta = -1$. We put $U_n = (\alpha^n - \beta^n)/(\alpha-\beta)$ $(n \ge 0)$. In particular, if $\beta = (1-\sqrt{5})/2$, then U_n $(n \ge 0)$ coincide with the Fibonacci numbers F_n $(n \ge 0)$. For $s \in \mathbb{N}$, set $\sigma_0(s) = 1$, and for $s \ge 2$, set

$$\sigma_i(s) = (-1)^i \sum_{1 \le r_1 < \ldots < r_i \le s-1} r_1^2 \cdots r_i^2 \quad (1 \le i \le s-1),$$

which are the elementary symmetric functions of the $s-1$ numbers $-1,\ -2^2,\ \ldots,\ -(s-1)^2$. Let a_j be the coefficients of $\operatorname{cosec}^2 x = x^{-2} + \sum_{j=0}^{\infty} a_j x^{2j}$, which are given by

$$a_{j-1} = (-1)^{j-1}(2j-1)2^{2j}B_{2j}/(2j)! \quad (j \ge 1),$$

where B_{2j} are the Bernoulli numbers.

Theorem 1. *Suppose that $\beta \in \overline{\mathbb{Q}}$, and set $\Phi_{2s} := (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty} U_n^{-2s}$ $(s \in \mathbb{N})$. Then the numbers $\Phi_2,\ \Phi_4,\ \Phi_6$ are algebraically independent, and for any integer $s \ge 4$*

$$\Phi_{2s} = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\mu_s - \sum_{j=1}^{s-1} \frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)(\varphi_j - (-1)^s\psi_j - a_j)\right)$$

with

$$\mu_s = \Phi_2 \quad (s \textit{ odd}), \quad = \frac{1}{3}\Big(4\Phi_2^2 + 2\Phi_2 - 18\Phi_4 + \omega - \frac{5}{4}\Big) \quad (s \textit{ even}),$$

$$\varphi_1 = \frac{4}{3}\Big(32\Phi_2^2 - 5\Phi_2 - \omega + \frac{13}{10}\Big), \quad \varphi_2 = -\frac{4}{63}(24\Phi_2 - 1)\Big(112\Phi_2^2 - 21\Phi_2 - 5\omega + \frac{77}{12}\Big),$$

$$\varphi_j = \frac{3}{(j-2)(2j+3)}\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1} \quad (j \ge 3),$$

$$\psi_1 = \frac{4}{3}\Big(16\Phi_2^2 - 13\Phi_2 - 5\omega + \frac{25}{4}\Big), \quad \psi_2 = \frac{4}{9}(24\Phi_2 - 1)\Big(16\Phi_2^2 - 13\Phi_2 - 5\omega + \frac{25}{4}\Big),$$

$$\psi_j = \frac{1}{j(2j-1)}\left(2(24\Phi_2 - 1)\psi_{j-1} - 3\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1}\right) \quad (j \ge 3),$$

where $\omega = (56\Phi_6 + 5/4)/(4\Phi_2 + 1)$.

Theorem 2. *Put $h_{2s} := (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty} U_{2n}^{-2s}$ $(s \in \mathbb{N})$. Then, for $-1 < \beta < -1 + \delta_0$,*

$$(\alpha^2 - \beta^2)^{2s} h_{2s} = \big(1 + O(e^{-(\pi^2/2)\eta^{-1}})\big)\sum_{j=0}^{\infty}\Gamma_j^{(s)}\eta^j, \quad \eta := -\log(-\beta),$$

where δ_0 is a sufficiently small positive number. The sum on the right-hand side is a convergent series with coefficients $\Gamma_j^{(s)} \in \mathbb{Q}[\pi]$. In particular

$$\Gamma_0^{(s)} = 2^{2s-1}(-1)^{s-1}B_{2s}\pi^{2s}/(2s)!\,.$$