Minkowski's second theorem over a Severi-Brauer variety

Takao Watanabe
Graduate School of Science, Osaka University
Toyonaka, Osaka, 560-0043 Japan
e-mail:watanabe@math.wani.osaka-u.ac.jp

Let $\boldsymbol{q}$ be a positive definite quadratic form on $\mathbf{R}^n$ of discriminant $\mathrm{disc}(\boldsymbol{q})$ and $\lambda_i(\boldsymbol{q})$, $i = 1, 2, \cdots, n$, the successive minima of $\boldsymbol{q}$, i.e.,

$$\lambda_i(\boldsymbol{q}) = \min\{\lambda > 0 \ : \ \{x \in \mathbf{Z}^n \ : \ \boldsymbol{q}(x) \le \lambda^2\} \text{ contains } i \text{ linearly independent vectors}\}.$$

Then Minkowski's second theorem asserts that the inequality

$$\lambda_1(\boldsymbol{q})\lambda_2(\boldsymbol{q}) \cdots \lambda_n(\boldsymbol{q}) \le \gamma_n^{n/2} \mathrm{disc}(\boldsymbol{q})^{1/2} \tag{1}$$

holds, where γ_n denotes *Hermite's constant*.

Some generalizations of Minkowski's second theorem were studied by Weyl, Mahler, Macfeat, Bombieri, Vaaler and Thunder,... etc. Vaaler recently extended (1) to a twisted height on a vector space over an algebraic number field. To state Vaaler's result, let k denote an algebraic number field and $\mathbf{A}$ the adele ring of k. For every n by n invertible matrix $g \in GL_n(\mathbf{A})$ with entries in $\mathbf{A}$, the twisted height H_g is defined on the n-dimensional vector space k^n. The successive minima $\lambda_i(g)$, $i = 1, 2, \cdots, n$, of g are defined by

$$\lambda_i(g) = \min\{\lambda > 0 \ : \ \{x \in k^n \ : \ H_g(x) \le \lambda\} \text{ contains } i \text{ linearly independent vectors}\}.$$

Then, Vaaler proved the inequality

$$\lambda_1(g)\lambda_2(g) \cdots \lambda_n(g) \le \gamma_n(k)^{n/2} |\det g|_{\mathbf{A}}^{1/[k:\mathbf{Q}]} . \tag{2}$$

Here the constant $\gamma_n(k)$ is the generalized Hermite constant of k defined by Icaza and Thunder. The inequality (2) coincides with (1) when $k = \mathbf{Q}$, $g = g_\infty$ (i.e., the finite adele part of g is the identity) and the symmetric matrix corresponding to $\boldsymbol{q}$ is equal to ${}^t g_\infty g_\infty$.

In my talk, I will show that Minkowski's second theorem is extended to a free module over the matrix algebra $\mathfrak{A} = M_m(D)$, where D is a central simple division algebra over a global field. We give a definition of the twisted heights on $\mathfrak{A}^n$ and introduce the generalized Hermite constant $\gamma_n(\mathfrak{A})$ of $\mathfrak{A}$. Then we obtain Minkowski's second theorem for the successive minima of a given twisted height. Our theorem recovers (2) when $m = 1$ and $D = k$. Since the twisted height H_g for $g \in GL_n(\mathbf{A})$ is indeed a height on the projective space $\mathbf{P}^{n-1}(k)$, the inequality (2) is regarded as a statement on $\mathbf{P}^{n-1}(k)$. In this point of view, our result may be considered as enlargement of a base space from a projective space to a Severi-Brauer variety.