SOME MEAN VALUE THEOREMS FOR THE SQUARE OF CLASS NUMBERS TIMES REGULATOR OF QUADRATIC EXTENSIONS

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We fix an algebraic number field k. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$, $\mathfrak{M}_{\mathrm{f}}$, $\mathfrak{M}_\mathbb{R}$ and $\mathfrak{M}_\mathbb{C}$ denote respectively the set of all places of k, all infinite places, all finite places, all real places and all complex places. For $v \in \mathfrak{M}$ let k_v denotes the completion of k at v and if $v \in \mathfrak{M}_{\mathrm{f}}$ then let q_v denote the order of the residue field of k_v. We let r_1, r_2, and e_k be respectively the number of real places, the number of complex places, and the number of roots of unity contained in k. We denote by $\zeta_k(s)$ the Dedekind zeta function of k.

To state the result, we classify quadratic extensions of k via the splitting type at places of $\mathfrak{M}_\infty$. Note that if $[F:k]=2$, then $F \otimes k_v$ is isomorphic to either $\mathbb{R}\times\mathbb{R}$ or $\mathbb{C}$ for $v \in \mathfrak{M}_\mathbb{R}$ and to $\mathbb{C}\times\mathbb{C}$ for $v \in \mathfrak{M}_\mathbb{C}$. We fix a $\mathfrak{M}_\infty$-tuples $L_\infty = (L_v)_{v\in\mathfrak{M}_\infty}$ where $L_v \in \{\mathbb{R}\times\mathbb{R}, \mathbb{C}\}$ for $v \in \mathfrak{M}_\mathbb{R}$ and $L_v = \mathbb{C}\times\mathbb{C}$ for $v \in \mathfrak{M}_\mathbb{C}$. We define

$$\mathfrak{Q}(L_\infty) = \{F \mid [F:k]=2, F\otimes k_v \cong L_v \text{ for all } v \in \mathfrak{M}_\infty\}.$$

Let $r_1(L_\infty)$ and $r_2(L_\infty)$ be the number of real places and complex places of $F \in \mathfrak{Q}(L_\infty)$, respectively. (This does not depend on the choice of F.) For $v \in \mathfrak{M}_{\mathrm{f}}$ we put

$$E_v = 1 - 3q_v^{-3} + 2q_v^{-4} + q_v^{-5} - q_v^{-6}, \quad E'_v = 2^{-1}(1-q_v^{-1})^3(1+2q_v^{-1}+4q_v^{-2}+2q_v^{-3}).$$

Theorem 1. *Let $n \geq 2$. We fix an $L_\infty = (L_v)_{v\in\mathfrak{M}_\infty}$ and $v_1, v_2, \dots, v_n \in \mathfrak{M}_{\mathrm{f}}$. Then we have*

$$\lim_{X\to\infty}\frac{1}{X^2}\sum_{\substack{F\in\mathfrak{Q}(L_\infty)\\ F:\text{not split at } v_1,\dots,v_n\\ |\Delta_{F/k}|\leq X}} h_F^2R_F^2 = \frac{(\mathrm{Res}_{s=1}\zeta_k(s))^3\Delta_k^2e_k^2\zeta_k(2)^2}{2^{r_1+r_2+1}2^{2r_1(L_\infty)}(2\pi)^{2r_2(L_\infty)}}\cdot\prod_{1\leq i\leq n}E'_{v_i}\prod_{\substack{v\in\mathfrak{M}_{\mathrm{f}}\\ v\neq v_1,\dots,v_n}}E_v.$$

Combined with the result of Kable-Yukie, we also obtain the limit of certain correlation coefficients. For simplicity we state in the case $k=\mathbb{Q}$, but similar statements are true for arbitrary number fields.

Theorem 2. *We fix a prime number l satisfying $l \equiv 1(4)$. For any quadratic field $F = \mathbb{Q}(\sqrt{m})$ other than $\mathbb{Q}(\sqrt{l})$, we put $F^* = \mathbb{Q}(\sqrt{ml})$. For a positive number X, we denote by $\mathcal{A}_l(X)$ the set of quadratic fields F such that $-X < D_F < 0$ and $F\otimes\mathbb{Q}_l$ is the quadratic unramified extension of $\mathbb{Q}_l$. Then we have*

$$\lim_{X\to\infty}\frac{\sum_{F\in\mathcal{A}_l(X)}h_Fh_{F^*}}{\left(\sum_{F\in\mathcal{A}_l(X)}h_F^2\right)^{1/2}\left(\sum_{F\in\mathcal{A}_l(X)}h_{F^*}^2\right)^{1/2}} = \prod_{\left(\frac{p}{l}\right)=-1}\left(1-\frac{2p^{-2}}{1+p^{-1}+p^{-2}-2p^{-3}+p^{-5}}\right),$$

where $\left(\frac{p}{l}\right)$ is the Legendre symbol and p runs through all the primes satisfying $\left(\frac{p}{l}\right)=-1$.

Our approach to prove these mean value theorems is the use of the theory of global zeta functions associated with prehomogeneous vector space $(\mathrm{GL}(2)\times\mathrm{GL}(2)\times\mathrm{GL}(2), \mathrm{Aff}^2\otimes\mathrm{Aff}^2\otimes\mathrm{Aff}^2)$ and its k-forms.

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