# ON THE ZEROS OF THE SYMMETRIC SQUARE $L$-FUNCTION ASSOCIATED WITH THE RAMANUJAN DELTA-FUNCTION

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**Abstract**

Let $\Delta(z)$ be the Ramanujan delta-function

$$\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}.$$

For each $p$ let $\alpha_p$ and $\beta_p$ be the roots of the polynomial $X^2-\tau(p)X+p^{11}$. Then the symmetric square $L$-function $D_\Delta(s)$ attached to $\Delta(z)$ is defined by the Euler product

$$D_\Delta(s) = \prod_p((1-\alpha_p^2p^{-s})(1-\alpha_p\beta_pp^{-s})(1-\beta_p^2p^{-s}))^{-1}.$$

The Euler product converges only when $\mathrm{Re}(s)$ is large, but there is a holomorphic continuation to the whole complex plane. It is well known that $D_\Delta(z)$ has the integral expression

$$\zeta^*(s)D_\Delta(s+11) = \frac{(4\pi)^{s+11}}{\Gamma(s+11)}\int_{PSL_2(\mathbf{Z})\backslash} y^{12}|\Delta(z)|^2E^*(z,s)\frac{dxdy}{y^2},$$

where $\zeta^*(s)$ is the completed Riemann zeta function and $E^*(z,s)$ is the completed non-holomorphic Eisenstein series for $PSL_2(\mathbf{Z})$. By using this integral expression and a property of the holomorphic projection of $\Delta(z)E^*(z,s)$, we obtain the decomposition

$$\tau(m)\zeta^*(s)D_\Delta(s+11) = \frac{(4\pi)^{s+11}}{m^{11}\Gamma(11)\Gamma(s+11)}\left\{C_m(s)+R_m(s)\right\}$$

for any $m\geq 1$, where

$$C_m(s) = \tau(m)\left\{\frac{\Gamma(s+11)}{(4\pi m)^{s+11}}\zeta^*(2s)+\frac{\Gamma(12-s)}{(4\pi m)^{12-s}}\zeta^*(2s-1)\right\}$$

and

$$R_m(s) = \frac{\Gamma(s+11)\Gamma(12-s)}{2\,(4\pi\sqrt{m}\,)^{11}}\sum_{\substack{n=-m+1\\ n\neq 0}}^{\infty}\frac{\tau(m+n)}{(m+n)^{11/2}}\,|n|^{s-1}\sigma_{1-2s}(|n|)P_{s-1}^{-11}\left(\frac{2m+n}{n}\right).$$

Now we denote by $R_m^{\mathrm{fin}}(s)$ a finite subseries of $R_m(s)$.

In this talk we show that all zeros of $C_m(s)+R_m^{\mathrm{fin}}(s)$ lie on the line $\mathrm{Re}(s)=1/2$ except for finite ones.