# Analytic Number Theory and Related Topics [*]

Date : October 15 (Tue) 10:15 – October 18 (Fri) 16:05, 2019
Place : Room 420, Research Institute for Mathematical Sciences (RIMS)
Kyoto University, Kyoto, JAPAN
Organizers : Masatoshi Suzuki (Tokyo Institute of Technology)
Takashi Nakamura (Tokyo University of Science)

## Abstracts

**October 15 (Tue)**

10:15 – 10:25     Opening

10:25 – 10:55     **Masahiro Mine** (Tokyo Institute of Technology)

"Discrete value-distributions for families of automorphic $L$-functions"

Ihara and Matsumoto studied certain density functions closely related to the value-distribution of Dirichlet $L$-functions and named them $M$-functions. In their study, the orthogonality of characters played a key role. In this talk, we consider the value-distributions of automorphic $L$-functions for some families of modular forms. We find that one can use the result on the equidistributions of eigenvalues of Hecke operators instead of the orthogonality of characters. The main result is to express various averages of values of automorphic $L$-functions as integrals of an automorphic analogue of Ihara–Matsumoto's $M$-function. This is also a generalization of the results of Cogdell–Michel, Fomenko, and Golubeva concerning the values of automorphic $L$-functions at $s = 1$.

11:10 – 11:50     **Masanori Katsurada** (Keio University)

"Asymptotic expansions associated with a non-holomorphic Eisenstein series of two complex variables" (joint work with Takumi Noda)

Let $\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2$ be complex variables, $z = x + iy \in \mathfrak{H}^+$ a complex parameter in the upper half-plane, and write $\langle \boldsymbol{s} \rangle = s_1 + s_2$. The non-holomorphic Eisenstein series $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; z)$ of two complex variables $\boldsymbol{s}$ is defined by

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; z) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m + nz)^{-s_1}(m + n\overline{z})^{-s_2} \qquad (\mathrm{Re}\langle \boldsymbol{s} \rangle > 2),$$

and its meromorphic continuation over the whole $\boldsymbol{s}$-space $\mathbb{C}^2$, where $\overline{z}$ denotes the complex conjugate of $z$, and the argument of each summand is chosen with $-\pi \leq \arg(m+nz) < \pi$ and $-\pi < \arg(m+n\overline{z}) \leq \pi$. Note that $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; z)$ reduces: i) if $\boldsymbol{s} = (s, 0)$ to the holomorphic Eisenstein series $F(s; z)$ of one complex variable; ii) if $\boldsymbol{s} = (s, s)$ to the classical Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s; z)$; and iii) if $\boldsymbol{s} = (s + k, s)$ for any $k \in 2\mathbb{Z}$ to $\zeta(2s)E_k(s; z)$, where $\zeta(s)$ and $E_k(s; z)$ denote the Riemann zeta-function and the Eisenstein series of weight $k$, respectively.

We shall show in this talk that complete asymptotic expansions exist for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; z)$ when $y = \operatorname{Im} z \to +\infty$ and when $z \to 0$ through the sector $\mathfrak{H}^+$. Several applications of our main formulae are also presented, for e.g., on: i) two variable analogues of the classical Kronecker limit formulae as $\boldsymbol{s} \to \boldsymbol{1} = (1,1)$ and at $\boldsymbol{s} = (0,0) = \boldsymbol{0}$; and ii) closed form evaluations of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; z)$ at any integer lattice point $\boldsymbol{s} \in \mathbb{Z}^2$.

13:30 – 14:00     **Junhyeong Kim** (Kyushu University)

"Leafwise-cohomological expression of dynamical zeta functions on foliated dynamical systems"

Arithmetic topology is a cross-sectional field which studies an analogy between number theory and 3-dimensional topology. In this talk, we mainly consider dynamical zeta function which is an analogy of Riemann zeta function in arithmetic topology. For the case of Riemann zeta function, C. Deninger and C. Soule showed that Riemann zeta function can be expressed in terms of the regularized-determinant and the non-trivial zeros. The main result of this talk is to express dynamical zeta function in terms of the regularized-determinant and the zeros which are eigenvalues of an infinitesimal generator of the dynamical system.

14:15 – 14:55     **Hiroki Takahasi** (Keio University)

"Large deviation principle for arithmetic mean of continued fraction digits"

Khinchin proved that the arithmetic mean of continued fraction digits of Lebesgue almost every irrational in $(0, 1)$ diverges to infinity. Hence, none of the limit theorem, such as weak and strong law of large numbers does not hold. Nevertheless, we show that the large deviation principle holds.

15:15 – 15:45     **Hirotaka Kobayashi** (Nagoya University)

"On a certain sum of derivatives of Dirichlet $L$-functions"

D.S. Shanks conjectured that $\zeta'(\frac{1}{2} + i\gamma)$ is positive real in the mean, where $\gamma$ is the ordinate of the zero of the Riemann $\zeta$-function. Concerning this, A. Fujii considered the sum of $\zeta'(\rho)$, where $\rho$ runs over the zeros of $\zeta(s)$. In 1994, he obtained an asymptotic formula of the sum, and it is approximated by positive real value. Moreover, all of the main terms are real. In this talk, we show that the Dirichlet $L$-functions have the same property.

16:00 – 16:40     **Ade Irma Suriajaya** (Kyushu University)

"Improved error estimate for the number of zeros of the derivatives of the Riemann zeta function" (joint work with Fan Ge)

Speiser in 1935 showed that the Riemann hypothesis is equivalent to the first derivative of the Riemann zeta function having no non-real zeros to the left of the critical line. This result shows a relation between the distribution of zeros of the Riemann zeta function and that of its derivatives. Berndt, Levinson, and Montgomery investigated the distribution of non-real zeros of the derivatives of the Riemann zeta function in various setups unconditionally, meanwhile Akatsuka gave sharper estimates on real part distribution and number of zeros under the truth of the Riemann hypothesis. The latter was further improved by Ge, for the first derivative case. I extended Akatsuka's result to higher order derivatives before the existence of Ge's result. Ge and I were later able to extend his idea to the case of higher order derivatives which gives the same order of error term as that in the case of

the Riemann zeta function itself. This error term bound for the number of zeros of the Riemann zeta function is due to Littlewood, proven back in 1924.

I have presented this result in Japan only in two local seminars and at a poster session in a zeta conference. I wish to introduce this result with necessary details to analytic number theory audience in Japan and I consider this RIMS Conference to be the most appropriate place to do so.

**October 16 (Wed)**

9:30 – 10:10    **Tomokazu Onozuka**  (Kyushu University)

"Sum formula and Ohno's relation for the multiple zeta functions"
(joint work with Minoru Hirose and Hideki Murahara)

The sum formula and Ohno's relation are well known relations among multiple zeta values. In this talk, we give its generalizations for the multiple zeta functions.

10:25 – 10:55    **Yoshitaka Sasaki**  (Osaka University of Health and Sport Sciences)

"On evaluations of multiple zeta values at non-positive integers"

Special values of the multiple zeta-function at non-positive integers depend on limiting processes except for some exceptions. Recently, Onozuka gave the asymptotic expansion of the multiple zeta-function at non-positive integers. In this talk, I mainly show that recurrence formulas for the coefficients of the asymptotic expansion. Moreover, multiple zeta values for several limiting processes at non-positive integers are also evaluated by using such formulas.

11:10 – 11:50    **Tatsushi Tanaka**  (Kyoto Sangyo University)

"Rooted tree maps for multiple zeta values and for multiple $L$-values"
(partly joint work with Noriko Wakabayashi)

We introduce rooted tree maps (RTMs for short) which are recently defined based on the Connes-Kreimer Hopf algebra of rooted trees. They induce a class of relations for multiple zeta values which is equivalent to the linear part of Kawashima relations. In a recent joint work (ongoing) with N. Wakabayashi, it is gradually revealed that RTMs can be extended to give a class of relations for multiple $L$-values.

13:30 – 14:00    **Shota Inoue**  (Nagoya University)

"A relation between the zero distribution of the Riemann zeta-function and a Dirichlet polynomial for the prime numbers"

The speaker recently showed a formula for the logarithm of the Riemann zeta-function and its iterated integrals. This formula is a modification of a hybrid formula due to Gonek, Hughes, and Keating. By using this formula, the speaker obtained some results on the relationship between prime numbers and the distribution of zeros, and on the value distribution of the Riemann zeta-function. The speaker will present these results in this talk.

14:15 – 14:45    **Kenta Endo**  (Nagoya University)

"On the value distribution of iterated integrals of the logarithm of the Riemann zeta-function"
(joint work with Shota Inoue)

In this talk, we consider the iterated integral of $\log \zeta(\sigma + it)$ over the vertical line for $1/2 \leq \sigma < 1$. It is a famous open problem whether or not the values of the Riemann zeta-function on the critical line is dense in the complex plane. We consider a problem of this type for the iterated integral of $\log \zeta(\sigma + it)$ and obtain a result that the values of this function is dense in the complex plane under the Riemann hypothesis. Especially, we prove the denseness of the values of $\int_0^t \log \zeta(1/2 + i\beta)d\beta$ assuming the Riemann hypothesis. Moreover, we show that, for any $m \geq 2$, the denseness of the values of the $m$-times iterated integral of $\log \zeta(\sigma + it)$ is equivalent to Riemann hypothesis.

15:05 – 15:45    **Takao Komatsu**  (Zhejiang Sci-Tech University)

"Some generalizations of harmonic numbers and their applications"

Harmonic numbers often appear in combinatorial theories. They are closely related with Stirling numbers and Bernoulli numbers. The infinite series involving harmonic numbers yield the values of zeta functions. In this talk, we consider some different types of q-generalizations of harmonic numbers and give several identities and applications.

16:00 – 16:50    **Shaofang Hong**  (Sichuan University)

"On the $p$-adic behaviors of Stirling numbers of the first and second kinds"

Stirling numbers are significant and common topics in number theory and combinatorics. Let $n$ and $k$ be positive integers such that $k \leq n$. The *Stirling number of the first kind*, denoted by $s(n, k)$, counts the number of permutations of $n$ elements with $k$ disjoint cycles. One can also characterize $s(n, k)$ by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} s(n,k)x^k.$$

The *Stirling number of the second kind* $S(n, k)$ is defined as the number of ways to partition a set of $n$ elements into exactly $k$ nonempty subsets. Thus

$$S(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n.$$

There are many authors who investigated the divisibility properties of Stirling numbers of the second kind. These authors include Lundell, Davis, Clarke, Lengyel, Chan, Kwong, Manna, Moll, Zhao, Wannemacker, Miska and the speaker et al. However, very few seems to be known about the divisibility properties of $s(n, k)$. It is known that the Stirling number of the first kind $s(n, k)$ is closely related to the $k$-th elementary symmetric function $H(n, k)$ of $1, 1/2, ..., 1/n$ by the following identity

$$s(n+1, k+1) = n!H(n, k),$$

where $H(n, 0) := 1$. In recent years, Lengyel, Komatsu, Young, Leonetti, Sanna, Adelberg, Qiu, Feng and the speaker made some progress in the study of the $p$-adic valuations of $s(n, k)$.

In this talk, we mainly report some old and new results and problems on the $p$-adic behaviors of Stirling numbers of both kinds.

17:30 – 19:30    Reception party at *Camphora* on the campus of Kyoto University

**October 17 (Thu)**

9:30 – 10:00 **Wataru Takeda** (Nagoya University)

"Brocard–Ramanujan problem for irreducble polynomials"

The Brocard–Ramanujan problem, which is an unsolved problem in number theory, is to find integer solutions $(x, l)$ of $x^2 - 1 = l!$. This problem still remains unsolved, on the other hand, it is known that there are many polynomials $P(x) \in \mathbf{Z}[x]$ such that $P(x) = l!$ has infinitely many solutions $(x, l)$. Recently, I studied integer solutions $(x, y, l)$ of an equation $F(x, y) = l!$, where $F$ is a homogeneous polynomial with integer coefficients. The aim of this talk to obtain the finiteness of $l$ such that $l!$ is represented as $F(x, y)$, where $F$ is irreducible or satisfies some condition. Moreover, we also consider an equation $F(x, y) = \Pi_K(l)$, where $\Pi_K$ is a generalized factorial function over number fields $K$. We also show that there exist number fields $K$ and polynomial $F(x, y)$ such that for infinitely many $l$ the factorial $\Pi_K(l)$ is represented as $F$.

10:15 – 10:55 **Yuta Suzuki** (Nagoya University)

"On even-odd amicable pairs"

A pair of positive integers $(A, B)$ is called an amicable pair if the sum of all proper divisors of $A$ equals $B$ and the sum of all proper divisors of $B$ equals $A$. It is a folklore conjecture that there is no even-odd amicable pair, i.e. an amicable pair consisting of even and odd numbers. It is easy to see that if there is an even-odd amicable pair $(A, B)$, then it should be of the form $(A, B) = (2^a M^2, N^2)$ with odd numbers $M, N$ up to permutation. This implies a trivial bound $A_1(X) \ll X^{\frac{1}{2}}$ for the number $A_1(X)$ of even-odd amicable pairs up to a given real number $X$. Pollack also remarked that the method of Iannucci and Luca provides a stronger bound $A_1(X) \ll X^{\frac{1}{2}}/(\log X)^{\frac{3}{2}+o(1)}$. In this talk, we prove a still stronger upper bound $A_1(X) \ll X^{\frac{1}{2}} \exp(-c(\log X \log \log \log X)^{\frac{1}{2}})$ with some constant $c > 0$.

11:10 – 11:50 **Koichi Kawada** (Iwate University)

"On sums of cubes of primes and an almost prime"
(joint work with Lilu Zhao)

Techniques developed by the end of 1930's possessed the ability to show, amongst others, that every sufficiently large odd integer is the sum of nine cubes of primes. Since then, the next target on the research of the Waring-Goldbach problem for cubes is to establish that every large even integer is the sum of eight cubes of primes. The currently existing technologies still appear inadequate to deliver the latter conclusion, but somewhat closer assertions are known. In fact, it was shown that every large even integer may be written as the sum of each of the following forms; (i) seven cubes of primes and a cube of $P_3$, (ii) six cubes of primes and two cubes of $P_2$. ($P_r$ means a natural number having at most $r$ prime factors, counted with multiplicity.) This talk shall report a recent refinement in this direction.

13:30 – 14:00 **Hiroaki Ito** (University of Tsukuba)

"Statistical properties of negative continued fractions"

From Birkhoff's ergodic theorem it is well known that Geometric mean etc. for regular continued fraction digits can be obtained. Let $N$ be a non-zero integer. An $N$-continued fraction is a modification of regular continued fraction having a fixed numerator $N$. Cor. Kraaikamp and others found an

invariant measure for $N$-continued fraction, but Birkhoff's theorem cannot be applied when $N = -1$, because its invariant measure is infinite. I talk about Khinchin's theorem and a property of Euclidean algorithm for $N = -1$.

14:15 – 15:05     **Sanoli Gun**   (The Institute of Mathematical Sciences)

"On zeros of modular forms"
(joint work with Joseph Oesterle)

F. K. C. Rankin and H. P. F. Swinnerton-Dyer showed that all zeros of $E_k$ in the standard fundamental domain $D$ are of modulus one. A result of Ghosh and Sarnak showed that "most" zeros in $D$ of a normalized Hecke eigenform are "real", i.e. they either lie on the imaginary axis or on the vertical line passing through $1/2$. In this context, we will report on some recent joint work with Joseph Oesterle.

15:25 – 15:55     **Seiji Kuga**   (Kyushu University)

"The locations of zeros of certain weakly holomorphic modular forms"
(joint work with Seiichi Hanamoto)

Let $p$ be one or a prime. When the genus of $\Gamma_0^+(p)$ is zero, we can construct a natural basis of the space of weakly holomorphic modular forms for $\Gamma_0^+(p)$. In this talk, we consider the location of the zeros on a fundamental domain of the elements of this basis.

16:10 – 16:50     **Eren Mehmet Kıral**   (Sophia Univesity)

"A parametrization of higher rank Kloosterman sums"
(joint work with Maki Nakasuji)

I will talk about our recent work with Professor Maki Nakasuji. The general Kloosterman sum on $SL_r$ corresponding to a Weyl group element is defined via an exponential sum over a $\Gamma_\infty$-double coset in the Bruhat cell of that Weyl group element. These sums come up naturally in the Fourier coefficients of general Poincaré series and hence in the general Bruggeman-Kuznetsov trace formula.

We give a nice and explicit parametrization for certain Bruhat cells (in $SL_3$ to start with), giving an explicit description of the relevant Kloosterman sums. We believe that such descriptions of these sums are vital for researchers in analytic number theory, especially if we want to recognize them for what they are when they arise in different contexts.

## October 18 (Fri)

9:30 – 10:00     **Haruki Ide**   (Keio University)

"Algebraic independence of the values of a certain entire function of two variables and its partial derivatives"

Some entire functions of one variable are known to have the notable property that their values and their derivatives of any order at nonzero distinct algebraic numbers are algebraically independent. For example, let $F(x) = \sum_{k=0}^{\infty} a^{R_k} x^k$, where $a$ is an algebraic number with $0 < |a| < 1$ and $\{R_k\}_{k \geq 0}$ is a certain linear recurrence including Fibonacci sequence. Tanaka proved that $F(x)$ has the property mentioned above, namely the infinite set $S_F := \{F^{(l)}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times}, \, l \geq 0\}$ is algebraically independent,

where $\overline{\mathbb{Q}}^{\times}$ denotes the set of nonzero algebraic numbers. He also proved that the infinite product $G(y) = \prod_{k=0}^{\infty}(1 - a^{R_k}y)$ has a similar algebraic independence property except its zeros, namely the infinite set $S_G := \{G^{(m)}(\beta) \mid \beta \in \overline{\mathbb{Q}}^{\times} \setminus \{a^{-R_k}\}_{k\geq 0}, \ m \geq 0\}$ is algebraically independent. The proof of the latter result is based on the relation $G'(y) = -G(y)H(y)$, where $H(y) = \sum_{k=0}^{\infty}(a^{R_k}/(1 - a^{R_k}y))$. Generalizing this relation, we will introduce in this talk a certain entire function of two variables having the following property: Both $S_F$ and $S_G$ mentioned above are included in the infinite set consisting of the values and the partial derivatives of any order of the two-variable function at any distinct algebraic points, and this infinite set is algebraically independent.

10:15 – 10:55      **Makoto Kawashima** (Osaka University)

"Linear independence of values of polylogarithm functions"
(joint work with Sinnou David and Noriko Hirata-Kohno)

This is a joint work with S. David (Sorbonne University) and N. Hirata-Kohno (Nihon University). Let $\mathrm{Li}_s(z)$ be the $s$-th polylogarithmic function with $s = 1, 2, \ldots$. Let $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$ be pairwise distinct rational numbers with $0 < \alpha_j < 1$, $(1 \leq j \leq m)$. In this talk, we prove a criterion for the linear independence over $\mathbb{Q}$ of the $rm + 1$ numbers : $\mathrm{Li}_1(\alpha_1), \mathrm{Li}_2(\alpha_1), \ldots, \mathrm{Li}_r(\alpha_1), \ldots, \mathrm{Li}_1(\alpha_m), \mathrm{Li}_2(\alpha_m), \ldots,$ $\mathrm{Li}_r(\alpha_m)$ and 1. We also show generalizations to an algebraic case with examples and an effective linear independence measure in both compex and $p$-adic field.

11:10 – 11:50      **Iekata Shiokawa** (Keio University)

"Irrationality exponents of certain alternating series"

Let $\{w_n\}$ be a sequence of positive integers, $\{y_n\}$ be a sequence of *nonzero* integers with $y_1 \geq 1$, and $m$ be a positive integer. We define $q_0 = 1$, $q_1 = w_0$, $q_{n+1} = q_{n-1}(w_n q_n^m + y_n)$ $(n \geq 1)$ and consider the series
$$\xi = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{y_1 y_2 \cdots y_n}{q_n q_{n-1}}.$$
We give exact values of the irrationality exponents $\mu(\xi)$ of the numbers $\xi$ under the condition that $\log|y_n| = o(\alpha^n)$, where $\alpha = (m + \sqrt{m^2 + 4})/2$. For example, we have $\mu(\xi) = 1 + \alpha$ for any $\{w_n\}$ satisfying $\sum_{k=0}^{\infty}(\log w_n)/\alpha^n < \infty$.

13:30 – 14:00      **Yusuke Tanuma** (Keio University)

"Algebraic independence of certain series related to integral parts of integral multiples of a real number"

Hecke-Mahler series is the generating function $h_{\omega}(z) = \sum_{k=1}^{\infty}[k\omega]z^k$ of the sequence $\{[k\omega]\}_{k\geq 1}$, where $\omega$ is a real number and $[\cdot]$ denotes the integral part. Let $\omega$ be a real quadratic irrational number. Nishioka proved that the infinite set $\{h_{\omega}^{(l)}(\alpha) \mid l \geq 0\}$, where $\alpha$ is a nonzero algebraic number inside the unit circle, is algebraically independent. On the other hand, Masser proved that the infinite set $\{h_{\omega}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}, \ 0 < |\alpha| < 1\}$ is algebraically independent. In this talk we study the algebraic independence of the "direct product" of these two sets. We also give similar results for exponential type Hecke-Mahler series $g_{\omega}(z) = \sum_{k=1}^{\infty}z^{[k\omega]}$.

14:15 – 15:05     **Wadim Zudilin**  (Radboud University Nijmegen)

"A method of creative microscoping"

Ramanujan's formulas for $1/\pi$ and their generalizations remain an amazing topic, with many mathematical challenges. Recently it was observed that the formulas possess spectacular 'supercongruence' counterparts. For example, truncating the sum in Ramanujan's formula

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}}(8k+1) = \frac{2\sqrt{3}}{\pi}$$

to the first $p$ terms correspond to the congruence

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}}(8k+1) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3}$$

valid for any prime $p > 3$. Some supercongruences were shown earlier through a tricky use of classical hypergeometric identities or the Wilf–Zeilberger method of creative telescoping. The particular example displayed above (and many other entries) were resistant to such techniques. In joint work with Victor Guo we develop a new method of 'creative microscoping' that simultaneously proves both the underlying Ramanujan's formula and its finite supercongruence counterparts. The main ingredient is an asymptotic analysis of $q$-analogues of Ramanujan's formulas at all roots of unity.

15:20 – 15:50     **Kota Saito**  (Nagoya University)

"On relations between Szemerédi's theorem and fractal dimensions of sets which do not contain weak arithmetic progressions"

In this talk, we give new lower and upper bounds for fractal dimensions of sets which do not contain $(k, \epsilon)$-arithmetic progressions (APs). More precisely, we say that a subset $F$ of real numbers does not contain $(k, \epsilon)$-APs if one cannot find any APs of length $k$ with gap difference $\Delta$ in the $\epsilon\Delta$-neighborhood of $F$. The goal of this talk is to show that upper and lower bounds for the Assouad and lower dimensions of sets which do not contain $(k, \epsilon)$-APs can be written in terms of $r_k(1/\epsilon)$. Here $r_k(N)$ denotes the largest cardinality of subsets of $\{1, \ldots, N\}$ which do not contain any APs of length $k$. Moreover, we find equivalent conditions between Szemerédi's theorem and bounds for fractal dimensions of sets which do not contain $(k, \epsilon)$-APs.

15:55     16:05     Closing